UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
ILLINOIS GEOMETRY LAB

FINDING INTEGERS FROM GROUP ORBITS

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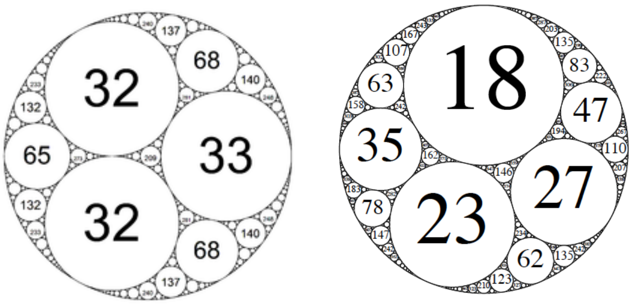
1 Introduction

The goal of this project is to study the local-global phenomenon arising from group orbits. It turns out that many arithmetic problems can be translated into studying the orbit of some special group. For example, the study of Apollonian circle packings. It is proved by Pappus that if we start with 4 mutually tangent circles with integer curvatures, then all the circles in the packing will have integer curvatures, which is known as an integral Apollonian circle packing. It is natural to ask what integers are the curvatures of Apollonian circle packings. This question in fact can be answered by studying the orbit of a thin subgroup of $\Gamma < SL_4(\mathbb{Z})$.



Another example comes a conjecture of Zaremba when studying the continued fraction expansions of fractions. In 1972, he conjectured that every natural number is the denominator of a reduced fraction whose partial quotients areabsolutely bounded. That is, there exists some absolute C > 1 so that for each d , there is some $(b,d) = 1$, so that

$$\frac{b}{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

with $\max a_j \le C$. This can also be turned into a problem of studying finitely generated subgroups Γ of $PSL_2(\mathbb{Z})$, where $\Gamma = \Gamma_{\mathscr{C}} =< \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} : a \in \mathscr{C} >$, where $\mathscr{C}$ is a set of finitely many integers.

In our setting, we study the orbits subgroups Γ of $GL_d(\mathbb{Z})$. Given a vector $v_0 \in \mathbb{Z}^d$, we are interested in the group orbit:

$$\mathscr{O} := \Gamma \cdot v_0$$

and the set of represented integers:

$$\mathscr{S} := < w_0, \mathscr{O} > \subset \mathbb{Z},$$

An easier set to study is called the *admissible set* defined as

$$\mathscr{A} = \{n \in \mathbb{Z} : n \in \mathscr{S} \pmod{q}\} \text{ for any } q \in \mathbb{N}.$$

The *local global conjecture* states that :

$$n \in \mathscr{A} \iff n \in \mathscr{S}$$

Strong Approximation Property of Γ implies that $\exists\ \mathscr{Z} \in \mathbb{Z}$, such that

$$n \in \mathscr{A} \iff n \in \mathscr{A}(mod \mathscr{Z}),$$

and we call $\mathscr{Z}$ the local obstruction. This allows us to find the admissible set by testing local obstructions at finitely many places.

2 Experiments

For this project, we want to study a weaker version of the local global conjecture, which is the *density* 1 *conjecture*. Denote

$$r_m := \frac{|\mathscr{S} \cap [-m, m]|}{|\mathscr{A} \cap [-m, m]|}$$

Then the *density* 1 *conjecture* says that

$$r_m \to 1, \text{ as } m \to \infty.$$

2.1 Subgroups of $SL_2(\mathbb{Z})$

We first consider subgroups of $SL_2(\mathbb{Z})$ and investigated the behavior of r_m in the following ways:

1 Fix a class of subgroup and vary w_0. It would be interesting to answer the following:

   - How does r_m change for various w_0?
   - How does r_m differ for groups that are conjugate?

We investigated $G_1 = \langle \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \rangle$ and $G_2 = \langle \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \rangle$. They are conjugate to each other in $SL_2(\mathbb{Z})$. For each group, we have 3 choices of w_0 to compare. The graph of r_m is shown in Figure 1 and Z denote the local obstruction. We notice that for most

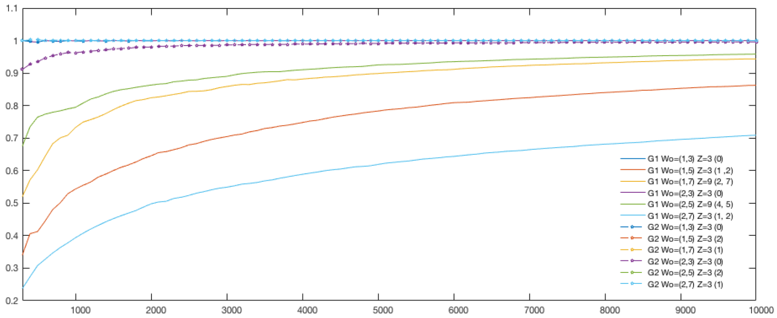


Figure 1: Graphs of r_m for $m \leq 10000$ of G_1 and G_2 with various w_0.

of the cases, $r_m \to 1$ as $m \to \infty$. Even though G_1 and G_2 are conjugate, the convergent rate of r_m is different for them and G_2 has a lower growth rate.

2 Different Groups with fixed w_0: We investigated 6 groups and the graph of r_m is shown in Figure 2. Due to the limitation of our algorithms, we compute r_m for m up to 40000. It can be seen that the behaviour differs from group to group very much.

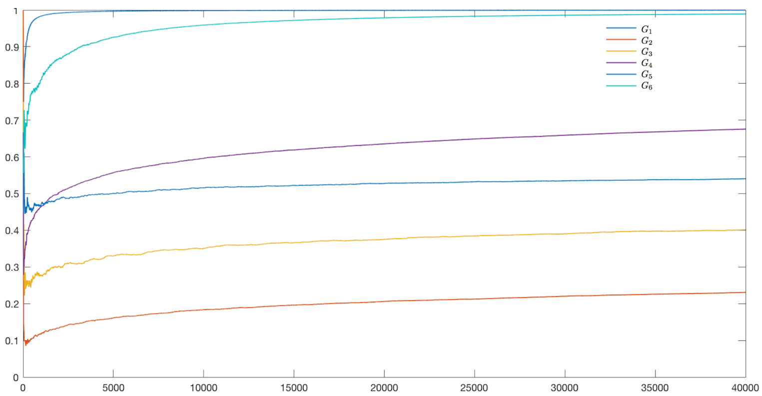


Figure 2: Graphs of r_m for $m \leq 40000$ of Different Groups with w_0 $(2,5)$

- For $G_1 =< \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} >$, $r_m \to 0.9996$, as $m \to 40000$.
- For $G_2 =< \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} >$, $r_m \to 0.2327$, as $m \to 40000$.

G1: $\delta \approx 0.7644$	G3: $\delta \approx 0.5391.$	$\delta \approx 0.4823.$
G2: $\delta \approx 0.5582$	G4: $\delta \approx 0.5709$	G6: $\delta \approx 0.5391$

Table 1: numerical data for ritical exponent

- For $G_3 =< \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} >$, $r_m \to 0.3995$, as $m \to 40000$.
- For $G_4 =< \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} >$, $r_m \to 0.674$, as $m \to 40000$.
- For $G_5 =< \begin{pmatrix} -2 & 5 \\ -3 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} >$, $r_m \to 0.5410$, as $m \to 40000$.
- For $G_6 =< \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} >$, $r_m \to 0.9903$, as $m \to 40000$.

2.2 Critical exponent

The differences of the limit of r_m is due to the different structure of these subgroups, especially the growth property of the group which is connected to the notion of *critical exponent.* The *Critical Exponent* δ of γ can be defined as follows: let

$$N(T) := \#\{\gamma \in \Gamma : \|\gamma\| \leq T\},$$

then it was known that

$$N(T) \sim C_\Gamma T^{2\delta}.$$

The critical exponent controls the rate of growth of $N(T)$. And it is believed that if $\delta \geq \frac{1}{2}$, then the local-global conjecture should hold. Based on the results of our experiment, the lower the δ is, the longer time it takes for r_m to approach 1, given $r_m \to 1$. This suggests that if the conjecture is true, it is difficult to prove the conjecture when δ is really small. We also computed the critical exponent for the groups we investigated above and the graph of $\frac{\log(N(T))}{2\log T}$ with respect to T is shown in Figure 3. And numerical data is in Table 1.

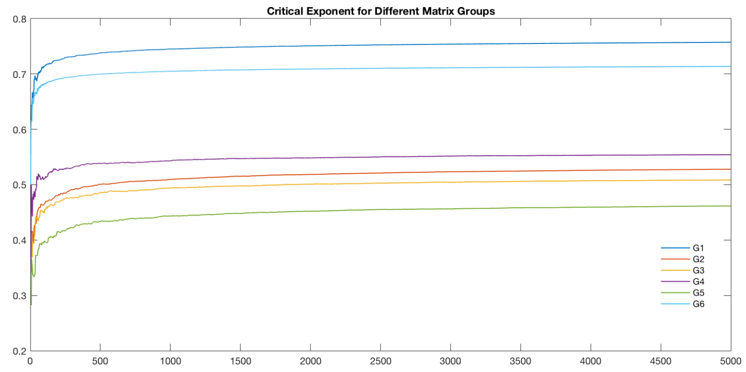


Figure 3: Critical exponent for different groups.

2.3 Subgroups of $SL_2(\mathbb{Z}[i])$

In the case of $SL_2(\mathbb{Z}[i])$, we experimented with two subgroups H_1

$$< \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3i & 1 \end{pmatrix} >$$

and H_2

$$< \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} > .$$

The integers are from the quadratic form

$$c_1^2 + c_2^2 + d_1^2 + d_2^2$$

for a complex matrix

$$\begin{pmatrix} a_1 i + a_2 & b_1 i + b_2 \\ c_1 i + c_2 & d_1 i + d_2 \end{pmatrix}.$$

We have the graphs for the density shown in Figure 4 and the critical exponents as shown in Figure 5

3 Future Directions

3.1 Theoretical

For further theoretical investigations, we would like to answer the following questions.

- For a fixed subgroup Γ, what is the connection between w_0 and r_m? How is the local obstruction related to the r_m?

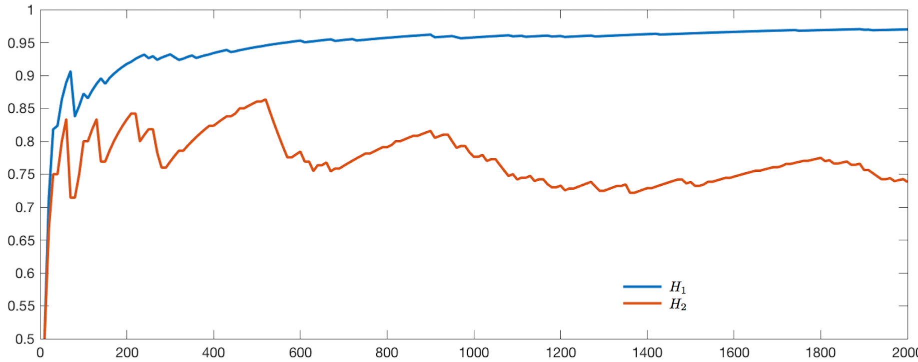


Figure 4: The density 1 conjecture for H_1 and H_2.

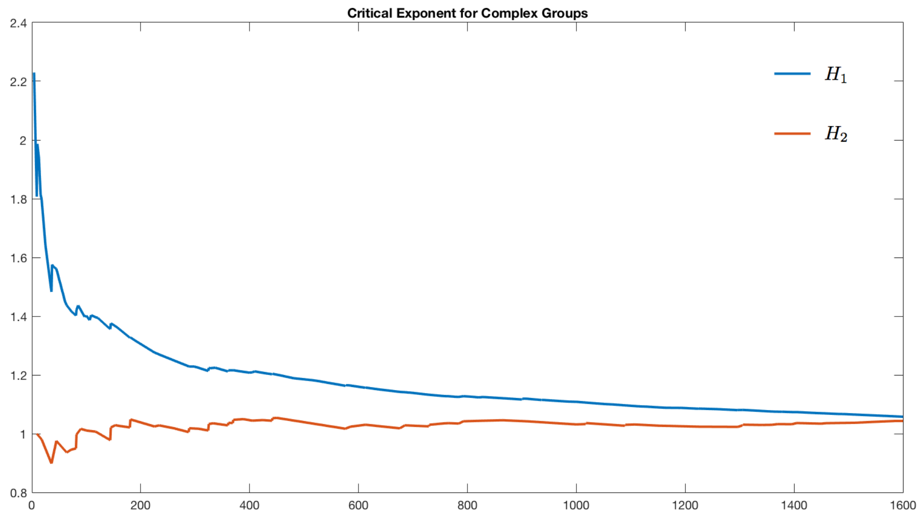


Figure 5: Critical Exponent for H_1 and H_2.

- For different subgroups, how does δ affect r_m?

3.2 Computational

Due to the limitation of the algorithm we used, we can't go very far, especially for the complex matrices. There are several computational aspects that we may consider.

- Improvement our computational algorithm to make it more efficient.
- Efficient memory use and large scale computation.
- Other algorithms of computing critical exponent of a subgroup.

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